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A Short Proof of Seymour's Characterization of the Matroids with the Max-Flow Min-Cut Property[1]

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Seymour proved that the set of odd circuits of a signed binary matroid (M, Σ) has the Max-Flow Min-Cut property if and only if it does not contain a minor isomorphic to $(M(K_4), E(K_4))$. We give a shorter proof of this result. © 2002 Elsevier Science (USA)

Key Words: Max-Flow Min-Cut; binary matroid; signed matroid; minor.

1. INTRODUCTION

The matroids considered in this paper are all binary. A *signed matroid* is a pair (M, Σ) where M is a matroid and $\Sigma \subseteq E(M)$. A subset X of elements of M is called *odd* (resp. *even*) if $|X \cap \Sigma|$ is odd (resp. even). We denote the set of odd circuits of (M, Σ) by $\mathscr{C}(M, \Sigma)$. We say that $\Sigma' \subseteq E(M)$ is a *signature* of (M, Σ) if $\mathscr{C}(M, \Sigma) = \mathscr{C}(M, \Sigma')$. Consider weights $w \in Z_+^{E(M)}$. We say that a subset $\mathscr{P}$ (with repetitions allowed) of $\mathscr{C}(M, \Sigma)$ is a *w-packing* (of odd circuits) if, for every element e of M, at most w_e circuits of $\mathscr{P}$ use e. A subset B of $E(M)$ is a *cover* of (M, Σ) if every odd circuit of (M, Σ) contains some element of B. It is straightforward to show that (inclusion-wise) minimal covers are signatures. Evidently, for every w-packing $\mathscr{P}$ and every cover B we must have $w(B) \geqslant |\mathscr{P}|$. If equality holds we say that (M, Σ) *packs* with respect to weights w. When $w_e = 1$ for all $e \in E(M)$ then a w-packing is called a *packing* and we say that (M, Σ) *packs* if it packs with respect to w. A signed matroid (M, Σ) has the *Max-Flow Min-Cut property* if it packs with respect to all non-negative integral weights w.

Let $e \in E(M)$. The *deletion* $(M, \Sigma)\backslash e$ of (M, Σ) is defined as $(M\backslash e, \Sigma - \{e\})$. The *contraction* $(M, \Sigma)/e$ of (M, Σ) is defined as follows: if $e \notin \Sigma$ then

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$(M, \Sigma)/e := (M/e, \Sigma)$; if $e \in \Sigma$ and e is not a loop then there exists a cocircuit D of M with $e \in D$ and $(M, \Sigma)/e := (M/e, \Sigma \triangle D)$. A *minor* of (M, Σ) is any signed matroid which can be obtained by a sequence of deletions and contractions. We say that (M, Σ) is *isomorphic* to (M', Σ') if after relabeling elements of M' we have $\mathscr{C}(M, \Sigma) = \mathscr{C}(M', \Sigma')$. It is easy to see that the Max-Flow Min-Cut property is closed under taking minors. We denote by $M(K_4)$ the graphic matroid of the complete graph on four vertices. Note that $(M(K_4), E(K_4))$ does not pack, as there are no two disjoint odd circuits and no edge intersects all odd circuits. Thus the essence of the next theorem is the "if" direction.

THEOREM 1.1 (Seymour [2]). *A signed binary matroid (M, Σ) has the Max-Flow Min-Cut property if and only if it has no minor isomorphic to $(M(K_4), E(K_4))$.*

2. A SHORT PROOF

Some of the ideas in this proof were used to give a characterization of evenly-bipartite graphs [1]. That paper also contains a proof for the graphic case of Seymour's theorem. The presentation of the proof follows closely the presentation of that proof.

*Proof of Theorem.*1.1 Let (M_0, Σ_0) be a minor-minimal signed binary matroid which does not have the Max-Flow Min-Cut property. Let e_0 be an element of M_0. Choose $w \in Z_+^{E(M_0)}$ such that (M_0, Σ_0) does not pack with respect to w and such that we first minimize $\Sigma_{e+e_0} w_e$ and then among all such w we maximize $w(e_0)$. Such a w exists since contracting e_0 in (M_0, Σ_0) is equivalent to setting w_{e_0} to a large value. Let (M, Σ) be obtained by replacing each element $f \in E(M)$ by w_f parallel elements (where f and its copies belong to Σ if and only if $f \in \Sigma_0$). Then (M, Σ) does not pack. Let e be one of the copies of e_0. Choose a set $\mathscr{F}$ of odd circuits of (M, Σ) such that:

(1) $\{C - \{e\} : C \in \mathscr{F}\}$ are disjoint.
(2) $|\mathscr{F}|$ is maximum with respect to (1).
(3) $|\{C : e \in C \in \mathscr{F}\}|$ is minimum with respect to (1) and (2).

Let $(\mathscr{F}_e, \mathscr{F}_{\bar{e}})$ be the partition of $\mathscr{F}$ into circuits containing e and not containing e, respectively.

CLAIM 1. $|\mathscr{F}_e| = 2$.

Proof. Choices (1) and (2) for $\mathscr{F}$ imply that $\{C - \{e\} : C \in \mathscr{F}\}$ is a maximum packing of $(M, \Sigma)/e$. Because of the choice of (M_0, Σ_0), $(M, \Sigma)/e$

packs. Thus there exists a cover B of $(M,\Sigma)/e$ with $|B| = |\mathscr{F}|$. Since B is also a cover of (M,Σ) and since (M,Σ) does not pack, $|\mathscr{F}_e| \geqslant 2$. Suppose $|\mathscr{F}_e| > 2$. Let (M_1,Σ_1) be obtained by adding an element e_1 parallel to e (with $\Sigma_1 = \Sigma$ if $e \notin \Sigma$ and $\Sigma_1 = \Sigma \cup \{e_1\}$ otherwise). By the choice of w, (M_1,Σ_1) packs and let $\mathscr{F}_1$, B_1 be the corresponding packing and cover with $|\mathscr{F}_1| = |B_1|$. Let $\mathscr{F}'$ be obtained by replacing the circuit C of $\mathscr{F}_1$ using e_1, by $C - \{e_1\} \cup \{e\}$. Since $2 = |\{C : e \in C \in \mathscr{F}'\}| < |\mathscr{F}_e|$, we must have $|\mathscr{F}'| < |\mathscr{F}|$. Thus $|B_1| < |\mathscr{F}|$. Since B_1 intersects all circuits of $\mathscr{F}, e \in B_1$. Since e_1 is parallel to e, $e_1 \in B_1$. But the packing consisting of the set of circuits of $\mathscr{F}_1$ avoiding e_1 together with the cover $B_1 - \{e_1\}$ imply that (M,Σ) pack, a contradiction. ∎

Let C_1, C_2 denote the circuits in $\mathscr{F}_e$. Note $C_1 \cap C_2 = \{e\}$. Recall that in a binary matroid, every cycle is the union of disjoint circuits.

Claim 2. *There are no odd cycles of* (M,Σ) *in* $C_1 \triangle C_2$.

Proof. Suppose for a contradiction, there exists an odd cycle $C \subseteq C_1 \triangle C_2$. Then C and $C_1 \triangle C_2 \triangle C$ contain odd circuits, say S and S', respectively. But then $\{S, S'\} \cup \mathscr{F}_{\bar{e}}$ contradicts choice (3) for $\mathscr{F}$ since $e \notin S \cup S'$. ∎

Let (M',Σ') be a minor of (M,Σ) where $E(M) - E(M') \subseteq C_1 \triangle C_2$. Let $C \subseteq C_1 \cup C_2$ be an odd circuit of (M',Σ'). We say that B is a *good cover* of C if it is a cover of (M',Σ') and $|B - C| = |\mathscr{F}_{\bar{e}}|$. We say that B is a *small cover* if it is a cover of (M',Σ') and $|B - \{e\}| = |\mathscr{F}_{\bar{e}}|$. Since circuits of $\mathscr{F}_{\bar{e}}$ are odd circuits of (M',Σ') it follows that if B is a good cover of C (resp. if B is a small cover) then $B - C$ (resp. $B - \{e\}$) intersects every odd circuit of $\mathscr{F}_{\bar{e}}$ exactly once.

Claim 3. *Every odd circuit* C *of* (M,Σ) *included in* $C_1 \cup C_2$ *has a good cover.*

Proof. Let $\mathscr{F}'$ be a maximum packing of $(M,\Sigma)\backslash C$. Observe that if $|\mathscr{F}'| > |\mathscr{F}_{\bar{e}}|$ then $\mathscr{F}' \cup \{C\}$ violates either choice (2) or (3) for $\mathscr{F}$. Thus $|\mathscr{F}'| = |\mathscr{F}_{\bar{e}}|$. It is easy to see that (M_0,Σ_0) has no odd loops. Hence, neither does (M,Σ) and $|C| \geqslant 2$. It follows from the choice of w that $(M,\Sigma)\backslash C$ packs. Hence there exists a cover B' of $(M,\Sigma)\backslash C$ with $|B'| = |\mathscr{F}_{\bar{e}}|$. Then $B := B' \cup C$ is a good cover of C. ∎

Claim 4. *For every element* $f \neq e$ *there exists a minimum cover* B *of* (M,Σ) *with* $f \in B$. *Moreover,* B *intersects every odd cycle included in* $C_1 \cup C_2$ *exactly once.*

Proof. Because of the choice of w, $(M,\Sigma)\backslash f$ packs and let $\mathscr{F}'$, B' be the corresponding packing and cover with $|\mathscr{F}'| = |B'|$. Since (M,Σ) does not pack, the size of the minimum cover of (M,Σ) is at least $|\mathscr{F}'| + 1 = |B'| + 1$. Thus $B := B' \cup \{f\}$ is a minimum cover of (M,Σ). Suppose for a contradiction there exists an odd cycle $C \subseteq C_1 \cup C_2$ with $|B \cap C| > 1$. As B is minimal, it is a signature, thus $|B \cap C|$ is odd, hence at least 3. Because B intersects all circuits of $\mathscr{F}_{\bar{e}}, |B| \geqslant |\mathscr{F}_{\bar{e}}| + 3 > |\mathscr{F}|$. Since $(M,\Sigma)/e$ packs and since $\{C - \{e\} : C \in \mathscr{F}\}$ is a maximum packing of $(M,\Sigma)/e$, minimum covers of $(M,\Sigma)/e$ have cardinality $|\mathscr{F}|$. Hence, minimum covers of (M,Σ) have cardinality at most $|\mathscr{F}|$. But then B is not a minimum cover, a contradiction. ∎

Let (M',Σ') be a minor of (M,Σ) which is minimal and satisfies the following properties:

(1) $E(M) - E(M') \subseteq C_1 \triangle C_2$.
(2) There exist odd cycles $C_1', C_2' \subseteq C_1 \cup C_2$ of (M',Σ') such that $\{e\} = C_1' \cap C_2'$.
(3) Every odd circuit $C \subseteq C_1' \cup C_2'$ of (M',Σ') has a good cover.
(4) (M',Σ') has no small cover.
(5) For all $f \in C_1' \triangle C_2'$, there exists $f' \in C_1' \triangle C_2'$ such that $\{f, f'\}$ intersects every odd cycle of (M',Σ') included in $C_1' \cup C_2'$ exactly once.

We claim that (M,Σ) satisfies properties (1)–(5). (1) is trivial; for (2) choose $C_1' = C_1$, $C_2' = C_2$; (3) holds because of Claim 3; (4) is satisfied since (M,Σ) does not pack. Let $f \in C_1 \triangle C_2$. Claim 4 implies that f is contained in a minimum cover B which intersects odd cycles included in $C_1 \cup C_2$ exactly once. Then C_1, C_2 imply that B contains exactly two elements in $C_1 \triangle C_2$. Hence (5) holds. Thus (M',Σ') is well defined.

Claim 5. *The only odd cycles of (M',Σ') included in $C_1' \cup C_2'$ are C_1' and C_2'.*

Proof.

Subclaim 5.1. *Each odd cycle in $C_1' \cup C_2'$ is a circuit. In particular C_1', C_2' are odd circuits.*

Proof. Otherwise there exists an odd cycle $C \subseteq C_1' \cup C_2'$ which is not a circuit. Partition C into an even cycle C_{even} and an odd circuit C_{odd}. Consider (5) and choose $f \in C_{even}$. Since C_{odd} is odd, we must have $f' \in C_{odd}$, but then $|\{f, f'\} \cap C| = 2$, a contradiction. ∎

Suppose for a contradiction, there exists an odd cycle $C \subseteq C_1' \cup C_2'$ distinct from C_1', C_2'. We know from Claim 2 that $e \in C$. Note that Subclaim 5.1 implies that C and $\bar{C} := C_1' \triangle C_2' \triangle C$ are odd circuits. Define

$$P_1 := C_1' \cap C - \{e\}, \qquad Q_1 := C_1' \cap \bar{C} - \{e\},$$
$$P_2 := C_2' \cap \bar{C} - \{e\}, \qquad Q_2 := C_2' \cap C - \{e\}.$$

Note $(P_1, Q_1, \{e\})$ partitions C_1' and $(P_2, Q_2, \{e\})$ partitions C_2'. Since $P_1 \cup P_2 = C_2' \triangle C$, $Q_1 \cup Q_2 = C_1' \triangle C$ it follows that $P_1 \cup P_2$ and $Q_1 \cup Q_2$ are even cycles. Odd circuits $C_1', C_2', C, \bar{C}$ imply the next result.

Subclaim 5.2. *For f, f' as in* (5) *either one element is in P_1 the other in P_2, or one is in Q_1 the other in Q_2.*

Subclaim 5.3. *Let S, S' be odd circuits of (M', Σ') which are included in $C_1' \cup C_2'$ and which are disjoint in $P_1 \cup P_2$. Then $P_1 \cup P_2 \subseteq S \cup S'$.*

Proof. Suppose for a contradiction $(P_1 \cup P_2) - (S \cup S') \neq \emptyset$. Consider f, f' as in (5) and choose $f \in P_i - S - S'$ for some $i \in \{1, 2\}$. Subclaim 5.2 implies $f' \in P_{3-i}$. Since S, S' are disjoint in $P_1 \cup P_2$, $\{f, f'\}$ intersects at most one of S, S', a contradiction. ∎

Case 1. There exist $P \subseteq P_1 \cup P_2$, $Q \subseteq Q_1 \cup Q_2$ such that $P \cup Q \cup \{e\}$ is an odd circuit of (M', Σ') such that for all covers B' of (M', Σ'), $|B' - P - \{e\}| > |\mathscr{F}_{\bar{e}}|$.

We may assume, after relabeling, that $P \cup Q \cup \{e\}$ corresponds to C_1'; that the odd cycles $(P \cup Q \cup \{e\}) \triangle C_1' \triangle C_2', (P \cup Q \cup \{e\}) \triangle (Q_1 \cup Q_2)$ correspond, respectively, to C_2' and C; and that $P_1 = P$, $Q_1 = Q$. Let $(M'', \Sigma'') := (M', \Sigma') \backslash P_1 / P_2$. We will show that (M'', Σ'') satisfies conditions (1)–(5) thereby contradicting the minimality of (M', Σ'). Clearly (1) holds. (2) is satisfied since $C_1'' := Q_1 \cup \{e\} = \bar{C} - P_2$ and $C_2'' := Q_2 \cup \{e\} = C_2' - P_2$ are odd cycles of (M'', Σ''). Let S be any odd circuit of (M'', Σ'') included in $C_1'' \cup C_2''$. Then there exists an odd circuit $S' \subseteq S \cup P_2$ in (M', Σ'). Since C_1' and S' are disjoint in $P_1 \cup P_2$, Subclaim 5.3 implies that $S' = S \cup P_2$. Subclaim 5.1 implies that $\bar{S} := (P_1 \cup P_2) \triangle S' = S \cup P_1$ is an odd circuit of (M', Σ'). From (3) we know that there exists a cover B' of (M', Σ') such that $|B' - \bar{S}| = |(B' - P_1) - S| = |\mathscr{F}_{\bar{e}}|$. Then $B' - P_1$ is a good cover of S in (M'', Σ''). Thus (3) holds. Note (4) holds by hypothesis (Case 1). Finally, (5) follows from Subclaim 5.2.

Case 2. For all $P \subseteq P_1 \cup P_2$, $Q \subseteq Q_1 \cup Q_2$ such that $P \cup Q \cup \{e\}$ is an odd circuit of (M', Σ') there exists a cover B' of (M', Σ') such that $|B' - P - \{e\}| = |\mathscr{F}_{\bar{e}}|$.

Let $(M'', \Sigma'') := (M', \Sigma')/(Q_1 \cup Q_2)$. We will show that (M'', Σ'') satisfies conditions (1)–(5) thereby contradicting the minimality of (M', Σ'). Clearly (1) holds. (2) is satisfied since $C''_1 := P_1 \cup \{e\} = C'_1 - Q_1$ and $C''_2 := P_2 \cup \{e\} = C'_2 - Q_2$ are odd cycles of (M'', Σ''). Consider an odd circuit included in $C''_1 \cup C''_2$ of (M'', Σ''). It is of the form $P \cup \{e\}$ where $P \subseteq P_1 \cup P_2$. Then there is an odd circuit $P \cup Q \cup \{e\}$ of (M', Σ') where $Q \subseteq Q_1 \cup Q_2$. By hypothesis (Case 2) there is a cover B' such that $|B' - P - \{e\}| = |\mathscr{F}_{\bar{e}}|$. Then B' is disjoint from $Q_1 \cup Q_2$. Hence, it is a good cover of $P \cup \{e\}$ in (M'', Σ''). Thus (3) holds. (4) is satisfied because (M'', Σ'') is a contraction minor of (M', Σ'). Finally, (5) follows from Subclaim 5.2. ∎

Let C'_1, C'_2 be the odd circuits given in (2). Let B_1, B_2 be the good covers of C'_1, C'_2 given by (3). Note $e \in B_1 \cap B_2$.

CLAIM 6. *There exists an odd circuit S of (M', Σ') such that $S \cap B_i \subseteq C'_i - \{e\}$ for $i = 1, 2$.*

Proof. Let $T := ((B_1 \cup B_2) - C'_1 - C'_2) \cup \{e\}$. Suppose for a contradiction, T is a cover. Then there exists a minimal cover $B \subseteq T$. For each $C \in \mathscr{F}_{\bar{e}}$ we have $|B \cap C| \leqslant 2$. Since B is a signature of (M', Σ'), $|B \cap C|$ is odd, hence equal to 1. But then B is a small cover, a contradiction to (5). Therefore, T is not a cover, i.e. $E(M) - T$ contains an odd circuit S. ∎

Choose S in Claim 6 so that $|S - C'_1 - C'_2|$ is minimized. Let $(M'', \Sigma'') := (M', \Sigma') \backslash (E(M') - C'_1 - C'_2 - S)$. For $i = 1, 2$, $B_i \cap C'_i$ is a cover of (M'', Σ'') which contains a minimal cover, say B'_i. Note $\{C'_1, C'_2, S\}$ implies that covers of (M'', Σ'') have cardinality at least two. Thus $|B'_i| \geqslant 2$ and since, $B'_i \subseteq C'_i$ and $|C'_i \cap B'_i|$ is odd, we have $|C'_i| \geqslant 3$.

CLAIM 7. *There are no two disjoint odd circuits in* (M'', Σ'').

Proof. Let $E' := S - C'_1 - C'_2$. Consider an odd circuit C of (M'', Σ'') distinct from C'_1, C'_2. Then Claim 5 implies $C \cap E' \neq \emptyset$. Suppose $e \notin C$. Then $E' \subseteq C$, for otherwise we would have chosen C instead of S. Moreover, (for $i = 1, 2$)$C \cap C'_i \neq \emptyset$ since $C \cap B'_i \neq \emptyset$. ∎

Since covers of (M'', Σ'') have cardinality at least two, (M'', Σ'') does not pack. Suppose for a contradiction that $(M, \Sigma) \neq (M'', \Sigma'')$. Let E' be the set of elements e' of (M, Σ) which are parallel to e and for which e' is in Σ if and only if $e \in \Sigma$. The choice of w implies that (M, Σ) can be obtained from (M'', Σ'') by adding elements in E' and that every element of (M'', Σ'') not in E' is in a cover of size two. Choose a cover of size two which contains an element of C_1 distinct from e. This cover does not contain e. This implies that (M, Σ) has a cover of size two. But then C_1, C_2 imply that (M, Σ) packs, a contradiction. Thus $(M, \Sigma) = (M'', \Sigma'')$. Since $(M, \Sigma)/f$ packs for all $f \in E(M)$ and since (M, Σ) has no two disjoint odd circuits, M has no parallel elements $f,\ f'$ where both $f,\ f' \in \Sigma$ or both $f,\ f' \notin \Sigma$. Hence, $w_f = 1$ for every $f \in E(M_0)$ and $(M_0, \Sigma_0) = (M, \Sigma)$. Therefore element e plays the same role as any other element of M. Thus for each $f \in E(M)$ we have odd circuits, say C_1^f, C_2^f, which intersect exactly in f. Define a graph G as follows: $V(G) := E(M)$ and $(f, f') \in E(G)$ if and only if $\{f, f'\}$ is a cover of (M, Σ).

Claim 8. *Edges of* G *form a perfect matching. Moreover, for each* $(f, f') \in E(G)$, $\{f, f'\} = E(M) - (C_1^f \triangle C_2^f)$.

Proof. Claim 4 and the fact that every element plays the same role implies that every element of (M, Σ) is in a cover of cardinality 2. Thus, every $f \in E(M)$ has degree at least one in G. Let f be any element of M. Odd circuits C_1^f, C_2^f imply (minimum covers are signatures) that all edges of G with endpoint f have an endpoint in $E(M) - \left(C_1^f \cup C_2^f\right)$, and conversely, all edges with an endpoint in $E(M) - \left(C_1^f \cup C_2^f\right)$ have endpoint f. It follows that for each $f \in E(M)$ all its neighbors in G have degree one. Therefore, edges of G form a perfect matching. Finally, let $f', f'' \in E(M) - \left(C_1^f \cup C_2^f\right)$, then both (f, f') and (f, f'') are edges of G. Since $E(G)$ is a matching $f' = f''$. Thus $E(M) - \left(C_1^f \cup C_2^f\right) = \{f'\}$. ∎

Let $e \in E(M)$. Recall $|C_1^e|,\ |C_2^e| \geqslant 3$. Claim 8 implies that we have elements $f, h \in C_1^e - \{e\}, f', h' \in C_2^e - \{e\}$, where $(f, f'), (h, h')$ are independent edges of G. It follows from Claim 8 that $\{f, h, f', h'\} = C_1^f \triangle C_2^f \triangle C_1^h \triangle C_2^h$. Thus $C_1^e \triangle \{f, h, f', h'\}$ is an odd cycle. It follows from Claim 5 that $C_1^e \triangle \{f, h, f', h'\} = C_2^e$. Hence, $|C_1^e| = |C_2^e| = 3$. Let $e' \in E(M)$ be such that $\{e, e'\}$ is a cover. Claim 8 implies that $E(M) = \{e, e', f, f', h, h'\}$. Then $C_1^{e'} = \{e', h, f'\}$ and $C_2^{e'} = \{e', f, h'\}$. Since $C_1^e, C_2^e, C_1^{e'}, C_2^{e'}$ are also covers of (M, Σ), the only odd circuits of (M, Σ) are $C_1^e, C_2^e, C_1^{e'}, C_2^{e'}$, i.e. (M, Σ) is isomorphic to $(M(K_4), E(K_4))$. ∎

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